

A Survey of Riccati Equation Results in Negative Imaginary Systems Theory and Quantum Control Theory

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Abstract—This paper presents a survey of some new applications of algebraic Riccati equations. In particular, the paper surveys some recent results on the use of algebraic Riccati equations in testing whether a system is negative imaginary and in synthesizing state feedback controllers which make the closed loop system negative imaginary. The paper also surveys the use of Riccati equation methods in the control of quantum linear systems including coherent H^∞ control.

I. INTRODUCTION

The algebraic Riccati equation has been applied in a range of applications in modern control theory including the linear quadratic regulator problem, the linear quadratic Gaussian problem and the H^∞ control problem; e.g., see [1]–[3]. In particular, it has played a major role in robust control theory; e.g., see [2], [4]. In this paper, we survey some results in which the Riccati equation plays a role in the emerging areas of negative imaginary systems theory and quantum linear systems theory.

The theory of negative imaginary systems is an emerging theory which is attracting interest among control theory researchers; e.g., see [5]–[17]. This theory is broadly applicable to problems of robust vibration control for flexible structures; e.g., see [5], [7], [18], [19]. Such flexible structures can be modelled by high order linear systems models with highly resonant dynamics [20]–[22]. A particular problem in the control of such systems is the fact that unmodelled spillover dynamics can severely degrade control system performance or lead to instability if the controller is not designed to be robust against this type of uncertainty; e.g., see [23], [24]. In addition, uncertainties in resonant frequencies and damping levels can cause similar problems of poor control system performance or instability. Negative imaginary systems theory provides a way of analyzing robustness and designing robust controllers for such flexible structures in the case of collocated force actuators and position sensors; e.g., see [18], [19], [25]–[32]. Although much of the theory of negative imaginary systems revolves around the use of linear matrix inequalities (LMIs), a number of recent results have emerged in which Riccati equations are used instead of LMIs; see [19].

Quantum feedback control systems have been an active area of research in recent years; e.g., see [33]–[38]. In particular, there has been considerable interest in the feedback control and modeling of linear quantum systems; e.g., see [35], [36], [36], [39]–[51]. Linear quantum systems commonly arise in the area of quantum optics; e.g., see [52]–[54]. Feedback control of quantum systems aims to achieve

closed loop properties such as stability [55], robustness [39] and entanglement [45], [56].

An important class of linear quantum stochastic models describe the Heisenberg evolution of the (canonical) position and momentum, or annihilation and creation operators of several independent open quantum harmonic oscillators that are coupled to external coherent bosonic fields; e.g., see [52], [51], [53], [57], [39]–[41], [44], [45], [48], [50], [58]–[60]. These linear stochastic models describe quantum optical devices such as optical cavities [54], [52], linear quantum amplifiers [53], and finite bandwidth squeezers [53]. In particular, we consider linear quantum stochastic differential equations driven by quantum Wiener processes; see [53]. Further details on quantum stochastic differential equations and quantum Wiener processes can be found in [61]–[63].

Some recent papers on the feedback control of linear quantum systems have considered the case in which the feedback controller itself is also a quantum system. Such feedback control is often referred to as coherent quantum control; e.g., see [36], [37], [39], [40], [42]–[44], [64]–[67]. In particular, some recent results on the coherent H^∞ control problem and the coherent LQG control problem involve the use of the Riccati equation and in this paper, we survey some of these results; e.g., see [39], [68], [69].

II. THE RICCATI EQUATION IN NEGATIVE IMAGINARY SYSTEMS THEORY

Negative imaginary (NI) systems theory is an emerging area of robust control theory which is concerned with the analysis and robust control of flexible systems with collocated force actuators and position sensors.

Definition 1: [5], [6], [70] A square transfer function matrix $M(s)$ is NI if the following conditions are satisfied:

- 1) $M(s)$ has no pole in $\text{Re}[s] > 0$.
- 2) For all $\omega > 0$ such that $s = j\omega$ is not a pole of $M(s)$,
$$j(M(j\omega) - M(j\omega)^*) \geq 0. \quad (1)$$
- 3) If $s = j\omega_0$ with $\omega_0 > 0$ is a pole of $M(s)$, then it is a simple pole and the residue matrix $K = \lim_{s \rightarrow j\omega_0} (s - j\omega_0)jM(s)$ is Hermitian and positive semidefinite.
- 4) If $s = 0$ is a pole of $M(s)$, then $\lim_{s \rightarrow 0} s^k M(s) = 0$ for all $k \geq 3$ and $\lim_{s \rightarrow 0} s^2 M(s)$ is Hermitian and positive semidefinite.

The following definition defines the strict negative imaginary property which is needed in the NI stability result.

Definition 2 (See [5].): A square real-rational proper transfer function matrix $N(s)$ is termed *strictly negative imaginary* (SNI) if

- 1) $N(s)$ has no poles in $\Re[s] \geq 0$;
- 2) $j[N(j\omega) - M^*(j\omega)] > 0$ for $\omega \in (0, \infty)$.

We now present the main stability result of negative imaginary systems theory that guarantees the robustness and stability of control systems involving the positive-feedback interconnection of an NI system and an SNI system; see also [5]–[7].

Theorem 1 ([5], [6]): Consider an NI transfer function matrix $M(s)$ with no poles at the origin and an SNI transfer function matrix $N(s)$, and suppose that $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$. Then, the positive-feedback interconnection of $M(s)$ and $N(s)$ is internally stable if and only if

$$\lambda_{\max}(M(0)N(0)) < 1. \quad (2)$$

An important result in the theory of negative imaginary systems is the following result which is referred to as the negative-imaginary lemma. This lemma can be used for testing if a given system is negative imaginary. It is also used in the proof of the above stability theorem.

Theorem 2 (Negative Imaginary Lemma, [5], [6], [19].): Let (A, B, C, D) be a minimal state-space realization of an $m \times m$ real-rational proper transfer function matrix $M(s)$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$. Then, $M(s)$ is NI if and only if $D = D^T$ and there exist matrices $P = P^T \geq 0$, $W \in \mathbb{R}^{m \times m}$, and $L \in \mathbb{R}^{m \times n}$ such that the following linear matrix inequality (LMI) is satisfied:

$$\begin{bmatrix} PA + A^T P & PB - A^T C^T \\ B^T P - CA & -(CB + B^T C^T) \end{bmatrix} = \begin{bmatrix} -L^T L & -L^T W \\ -W^T L & -W^T W \end{bmatrix} \leq 0. \quad (3)$$

The following result is our first result in the theory of negative imaginary systems which uses the Riccati equation. This result is an alternative version of the negative imaginary lemma which uses the Riccati equation rather than the LMI (3).

Theorem 3 ([19]): Suppose the transfer function matrix $M(s)$ has a minimal realization $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ such that $CB + B^T C^T > 0$. Then $M(s)$ is NI if and only if $D = D^T$ and there exists a matrix $P \geq 0$ that solves the algebraic Riccati equation

$$PA_0 + A_0^T P + PBR^{-1}B^T P + Q = 0 \quad (4)$$

where

$$\begin{aligned} A_0 &= A - BR^{-1}CA, \\ R &= CB + B^T C^T, \text{ and} \\ Q &= A^T C^T R^{-1}CA. \end{aligned}$$

In our next result, we use an algebraic Riccati equation to synthesize a state feedback controller such that the closed

loop system in NI. Consider the following state space representation for a linear uncertain system given as follows;

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x, \\ W(s) &= \Delta(s)Z(s), \end{aligned} \quad (5)$$

where we assume that $W(s)$ and $Z(s)$ are the Laplace transforms of the signals $w(t)$ and $z(t)$.

Here, $A \in \mathbb{R}^{n \times n}$, $B_1 \in \mathbb{R}^{n \times m}$, $B_2 \in \mathbb{R}^{n \times r}$, $C_1 \in \mathbb{R}^{m \times n}$, and $\Delta(s)$ is an strictly negative imaginary uncertainty transfer function matrix; e.g., see [5], [70]. Also, suppose that $K \in \mathbb{R}^{r \times n}$ is a state feedback matrix such that $u = Kx$. Then the closed-loop interconnection of the system (5) with the state feedback control law is given by

$$\begin{aligned} \dot{x} &= (A + B_2 K)x + B_1 w, \\ z &= C_1 x. \end{aligned} \quad (6)$$

and

$$W(s) = \Delta(s)Z(s). \quad (7)$$

Our aim is to construct the matrix K such that the corresponding closed-loop system (6) is stable and satisfies the NI property. From this, it follows that the closed-loop uncertain system will be stable for any SNI uncertainty $\Delta(s)$ [5], [7] providing the corresponding DC gain condition (2) is satisfied.

Consider the following Schur transformation:

$$A_f = U^T(A - B_2(C_1 B_2)^{-1}C_1 A)U = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad (8)$$

$$B_f = U^T(B_2(C_1 B_2)^{-1} - B_1 R^{-1}) = \begin{bmatrix} B_{f1} \\ B_{f2} \end{bmatrix}, \quad (9)$$

$$\tilde{B}_1 = U^T B_1 = \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix}, \quad (10)$$

where U is a unitary matrix.

The transformation (8) can be constructed such that A_{11} has all of its eigenvalues in the closed left half of the complex plane and A_{22} has all of its eigenvalues in the open right half of the complex plane; i.e., A_{22} is an anti-stable matrix.

Theorem 4 ([19]): Given an uncertain system (5) with $C_1 B_2$ non-singular and $R = C_1 B_1 + B_1^T C_1^T > 0$, define A_f , B_f , \tilde{B}_1 , U , A_{11} , A_{12} , A_{22} , B_{f1} , B_{f2} , B_{11} and B_{22} as in (8)–(10), where A_{22} is the anti-stable block of the A_f matrix. Then there exists a static state-feedback matrix K such that the closed-loop system (6) is NI if there exist matrices $T \geq 0$ and $S \geq 0$ such that

$$-A_{22}T - TA_{22}^T + B_{f2}RB_{f2}^T = 0, \quad (11)$$

$$-A_{22}S - SA_{22}^T + B_{22}R^{-1}B_{22}^T = 0. \quad (12)$$

and $T - S > 0$. Furthermore, a static state-feedback matrix K which makes the closed-loop system (6) NI and stabilizes the anti-stable matrix A_{22} is given by

$$K = (C_1 B_2)^{-1}(B_1^T P - C_1 A - R(B_2^T C_1^T)^{-1}B_2^T P), \quad (13)$$

where $P = UP_fU^T$ and $P_f = \begin{bmatrix} 0 & 0 \\ 0 & (T-S)^{-1} \end{bmatrix} \geq 0$ satisfies the algebraic Riccati equation

$$P_f A_f + A_f^T P_f - P_f B_f R B_f^T P_f + P_f \tilde{B}_1 R^{-1} \tilde{B}_1^T P_f = 0. \quad (14)$$

III. THE RICCATI EQUATION IN LINEAR QUANTUM SYSTEMS THEORY

We consider a class of linear quantum system models. These linear quantum system models take the form of quantum stochastic differential equations (QSDEs) which are derived from the quantum harmonic oscillator; e.g., see [39], [68], [69]. We will survey a number of results involving Riccati equations which arise in the theory of such quantum linear systems.

A. Quantum Harmonic Oscillators

We begin by considering a collection of n independent quantum harmonic oscillators which are defined on a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n, \mathbb{C})$; e.g., see [50], [62], [71]. Elements of the Hilbert space \mathcal{H} , $\psi(x)$ are the standard complex valued wave functions arising in quantum mechanics where x is a spatial variable. Corresponding to this collection of harmonic oscillators is a vector of *annihilation operators* $a = [a_1 \ a_2 \ \dots \ a_n]^T$. The adjoint of the operator a_i is denoted a_i^* and is referred to as a creation operator.

The quantum harmonic oscillators described above are assumed to be coupled to m external independent quantum fields modelled by bosonic annihilation field operators $\mathcal{A}_1(t), \mathcal{A}_2(t), \dots, \mathcal{A}_m(t)$ which are defined on separate Fock spaces \mathcal{F}_i defined over $L^2(\mathbb{R})$ for each field operator [56], [61]–[63]. For each annihilation field operator $\mathcal{A}_j(t)$, there is a corresponding creation field operator $\mathcal{A}_j^*(t)$, which is defined on the same Fock space and is the operator adjoint of $\mathcal{A}_j(t)$. The field annihilation operators are also collected into a vector of operators defined as follows: $\mathcal{A} = [\mathcal{A}_1 \ \mathcal{A}_2 \ \dots \ \mathcal{A}_m]^T$.

In order to describe the joint evolution of the quantum harmonic oscillators and quantum fields, we first specify the *Hamiltonian operator* for the quantum system which is a Hermitian operator on \mathcal{H} of the form

$$\mathbf{H} = \frac{1}{2} \begin{bmatrix} a^\dagger & a^T \end{bmatrix} M \begin{bmatrix} a \\ a^\# \end{bmatrix} \quad (15)$$

where $M \in \mathbb{C}^{2n \times 2n}$ is a Hermitian matrix of the form

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^\# & M_1^\# \end{bmatrix} \quad (16)$$

and $M_1 = M_1^\dagger$, $M_2 = M_2^T$. Also, we specify the *coupling operator* for the quantum system to be an operator of the form

$$L = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} \quad (17)$$

where $N_1 \in \mathbb{C}^{m \times n}$ and $N_2 \in \mathbb{C}^{m \times n}$. Also, we write

$$\begin{bmatrix} L \\ L^\# \end{bmatrix} = N \begin{bmatrix} a \\ a^\# \end{bmatrix} = \begin{bmatrix} N_1 & N_2 \\ N_2^\# & N_1^\# \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix}.$$

In addition, we define a *scattering matrix* which is a unitary matrix $S \in \mathbb{C}^{n \times n}$.

The Heisenberg evolution of the operator vectors a , and a^\dagger is described the following QSDEs (e.g., see equations (1) and (2) in [72] and equations (7) and (9) in [47]):

$$\begin{aligned} d \begin{bmatrix} a \\ a^\# \end{bmatrix} = & -i \left[\begin{bmatrix} a \\ a^\# \end{bmatrix}, \mathbf{H} \right] dt + \frac{1}{2} \left(L^\dagger \left[\begin{bmatrix} a \\ a^\# \end{bmatrix}, L^T \right]^T \right)^T dt \\ & + \frac{1}{2} \left[L^\#, \begin{bmatrix} a \\ a^\# \end{bmatrix}^T \right]^T L dt + \left[\begin{bmatrix} a \\ a^\# \end{bmatrix}, L^T \right] d\mathcal{A}^\# \\ & - \left[\begin{bmatrix} a \\ a^\# \end{bmatrix}, L^\dagger \right] d\mathcal{A}. \end{aligned} \quad (18)$$

Using (15) and (17), these equations then lead to the following QSDEs, which describe the dynamics of the quantum system under consideration:

$$\begin{aligned} \begin{bmatrix} da(t) \\ da(t)^\# \end{bmatrix} &= F \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt + G \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}; \\ \begin{bmatrix} d\mathcal{A}^{out}(t) \\ d\mathcal{A}^{out}(t)^\# \end{bmatrix} &= H \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt + K \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}, \end{aligned} \quad (19)$$

where

$$\begin{aligned} F &= \begin{bmatrix} F_1 & F_2 \\ F_2^\# & F_1^\# \end{bmatrix}; \quad G = \begin{bmatrix} G_1 & G_2 \\ G_2^\# & G_1^\# \end{bmatrix}; \\ H &= \begin{bmatrix} H_1 & H_2 \\ H_2^\# & H_1^\# \end{bmatrix}; \quad K = \begin{bmatrix} K_1 & K_2 \\ K_2^\# & K_1^\# \end{bmatrix}. \end{aligned} \quad (20)$$

Here,

$$\begin{aligned} F &= -iJM - \frac{1}{2}JN^\dagger JN; \\ G &= -JN^\dagger \begin{bmatrix} S & 0 \\ 0 & -S^\# \end{bmatrix}; \\ H &= N; \\ K &= \begin{bmatrix} S & 0 \\ 0 & S^\# \end{bmatrix}. \end{aligned} \quad (21)$$

The QSDEs (19), (20) define the general class of linear quantum systems considered. Such quantum systems can be used to model a large range of devices and networks of devices arising in the area of quantum optics including optical cavities, squeezers, optical parametric amplifiers, cavity QED systems, beam splitters, and phase shifters; e.g., see [35]–[37], [39], [44], [46], [48], [51]–[54], [67].

B. Coherent Quantum H^∞ Control

We formulate a coherent quantum control problem in which a linear quantum system is controlled by a feedback controller which is itself a linear quantum system. Such a controller is said to be physically realizable; e.g., see [39], [68], [69]. The fact that the controller is to be a quantum system means that any controller synthesis method needs to produce controllers which are physically realizable. The

problem we consider is the quantum H^∞ control problem in which it is desired to design a coherent controller such that the resulting closed loop quantum system is stable and attenuates specified disturbances acting on the system; see [39], [43]. In the standard quantum H^∞ control problem such as considered in [39], [43], the quantum noises are averaged out and only the external disturbance is considered. We now formulate the coherent quantum H^∞ control problem for a general class of quantum systems of the form (19), (20).

We consider quantum *plants* described by linear complex quantum stochastic models of the following form defined in an analogous way to the QSDEs (19), (20):

$$\begin{aligned} \begin{bmatrix} da(t) \\ da(t)^\# \end{bmatrix} &= F \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt \\ &+ \begin{bmatrix} G_0 & G_1 & G_2 \end{bmatrix} \begin{bmatrix} dv(t) \\ dw(t) \\ du(t) \end{bmatrix}; \\ dz(t) &= H_1 \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt + K_{12} du(t); \\ dy(t) &= H_2 \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt \\ &+ \begin{bmatrix} K_{20} & K_{21} & 0 \end{bmatrix} \begin{bmatrix} dv(t) \\ dw(t) \\ du(t) \end{bmatrix} \end{aligned} \quad (22)$$

where all of the matrices in these QSDEs have a form as in (20). Here, the input

$$dw(t) = \begin{bmatrix} \beta_w(t)dt + d\mathcal{A}(t) \\ \beta_w^\#(t)dt + d\mathcal{A}(t)^\# \end{bmatrix}$$

represents a disturbance signal where $\beta_w(t)$ is an adapted process; see [39], [42], [62]. The signal $u(t)$ is a control input of the form

$$du(t) = \begin{bmatrix} \beta_u(t)dt + d\mathcal{B}(t) \\ \beta_u^\#(t)dt + d\mathcal{B}(t)^\# \end{bmatrix}$$

where $\beta_u(t)$ is an adapted process. The quantity

$$dv(t) = \begin{bmatrix} d\mathcal{C}(t) \\ d\mathcal{C}(t)^\# \end{bmatrix}$$

represents any additional quantum noise in the plant. The quantities $\begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}$, $\begin{bmatrix} d\mathcal{B}(t) \\ d\mathcal{B}(t)^\# \end{bmatrix}$ and $\begin{bmatrix} d\mathcal{C}(t) \\ d\mathcal{C}(t)^\# \end{bmatrix}$ are quantum noises.

In the coherent quantum H^∞ control problem, we consider controllers which are described by QSDEs of the form (19),

(20) as follows:

$$\begin{aligned} \begin{bmatrix} da_c(t) \\ da_c(t)^\# \end{bmatrix} &= F_c \begin{bmatrix} a_c(t) \\ a_c(t)^\# \end{bmatrix} dt \\ &+ \begin{bmatrix} \bar{G}_c & \hat{G}_c & G_c \end{bmatrix} \begin{bmatrix} d\bar{w}_c \\ d\hat{w}_c \\ dy \end{bmatrix} \\ \begin{bmatrix} du(t) \\ d\hat{u}(t) \\ d\bar{u}(t) \end{bmatrix} &= \begin{bmatrix} H_c \\ \hat{H}_c \\ \bar{H}_c \end{bmatrix} \begin{bmatrix} a_c(t) \\ a_c(t)^\# \end{bmatrix} dt \\ &+ \begin{bmatrix} K_c & 0 & 0 \\ 0 & \bar{K}_c & 0 \\ 0 & 0 & \bar{K}_c \end{bmatrix} \begin{bmatrix} d\bar{w}_c \\ d\hat{w}_c \\ dy \end{bmatrix} \end{aligned} \quad (23)$$

where all of the matrices in these QSDEs have a form as in (20). Here the quantities

$$d\bar{w}_c = \begin{bmatrix} d\mathcal{A}_c(t) \\ d\mathcal{A}_c(t)^\# \end{bmatrix}, \quad d\hat{w}_c = \begin{bmatrix} d\mathcal{B}_c(t) \\ d\mathcal{B}_c(t)^\# \end{bmatrix}$$

are controller quantum noises. Also, the outputs du_0 and du_1 are unused outputs of the controller which have been included so that the controller can be physically realizable. Corresponding to the plant (22) and (23), we form the closed loop quantum system by identifying the output of the plant dy with the input to the controller dy , and identifying the output of the controller du with the input to the plant du . This leads to the following closed-loop QSDEs:

$$\begin{aligned} d\eta(t) &= \begin{bmatrix} F & G_2 H_c \\ G_c H_2 & F_c \end{bmatrix} \eta(t) dt \\ &+ \begin{bmatrix} G_0 & G_2 & 0 \\ G_c K_{20} & \bar{G}_c & \hat{G}_c \end{bmatrix} \begin{bmatrix} dv(t) \\ d\bar{w}_c(t) \\ d\hat{w}_c(t) \end{bmatrix} \\ &+ \begin{bmatrix} G_1 \\ G_c K_{21} \end{bmatrix} dw(t); \\ dz(t) &= \begin{bmatrix} H_1 & K_{12} H_c \end{bmatrix} \eta(t) dt \\ &+ \begin{bmatrix} 0 & K_{12} & 0 \end{bmatrix} \begin{bmatrix} dv(t) \\ d\bar{w}_c(t) \\ d\hat{w}_c(t) \end{bmatrix} \end{aligned} \quad (24)$$

where

$$\eta(t) = \begin{bmatrix} a(t) \\ a(t)^\# \\ a_c(t) \\ a_c(t)^\# \end{bmatrix}.$$

For a given quantum plant of the form (22), the coherent quantum H^∞ control problem involves finding a physically realizable quantum controller (23) such that the resulting closed loop system (24) is such that the following conditions are satisfied:

(i) The matrix

$$F_{cl} = \begin{bmatrix} F & G_2 H_c \\ G_c H_2 & F_c \end{bmatrix} \quad (25)$$

is Hurwitz;

(ii) The closed loop transfer function

$$\Gamma_{cl}(s) = H_{cl}(sI - F_{cl})^{-1} G_{cl}$$

satisfies

$$\|\Gamma_{cl}(s)\|_{\infty} < 1 \quad (26)$$

where

$$H_{cl} = \begin{bmatrix} H_1 & K_{12}H_c \end{bmatrix}, \quad G_{cl} = \begin{bmatrix} G_1 \\ G_c K_{21} \end{bmatrix}.$$

Theorem 5: (See also [3], [39].) Suppose that the plant (22) satisfies the following conditions:

- (i) $E_1 = K_{12}^{\dagger} K_{12} > 0$;
- (ii) $E_2 = K_{21} K_{21}^{\dagger} > 0$;
- (iii) The matrix $\begin{bmatrix} F - i\omega I & G_2 \\ H_1 & K_{12} \end{bmatrix}$ is full rank for all ω ;
- (iv) The matrix $\begin{bmatrix} F - i\omega I & G_1 \\ H_2 & K_{21} \end{bmatrix}$ is full rank for all ω .

Then the above coherent H^{∞} control problem has a solution if and only if the Riccati equations

$$\begin{aligned} (F - G_2 E_1^{-1} K_{12}^{\dagger} H_1)^{\dagger} X + X(F - G_2 E_1^{-1} K_{12}^{\dagger} H_1) + \\ X(G_1 G_1^{\dagger} - G_2 E_1^{-1} G_2^{\dagger}) X + \\ H_1^{\dagger} (I - K_{12} E_1^{-1} K_{12}^{\dagger}) H_1 = 0; \end{aligned} \quad (27)$$

$$\begin{aligned} (F - G_1 K_{21}^{\dagger} E_2^{-1} H_2) Y + Y(F - G_1 K_{21}^{\dagger} E_2^{-1} H_2) + \\ Y(H_1^{\dagger} H_1 - H_2^{\dagger} E_2^{-1} H_2) Y + \\ G_1 (I - K_{21}^{\dagger} E_2^{-1} K_{21}) G_1^{\dagger} = 0. \end{aligned} \quad (28)$$

have positive-semidefinite stabilizing solutions X and Y such that the spectral radius of XY is strictly less than one. In this case, the matrices F_c , G_c , and H_c in the controller system (23) can be constructed according to the formulas:

$$\begin{aligned} F_c &= F + G_2 H_c - G_c H_2 + (G_1 - G_c K_{21}) G_1^{\dagger} X; \\ G_c &= (I - YX)^{-1} (Y H_2^{\dagger} + G_1 K_{21}^{\dagger}) E_2^{-1}; \\ H_c &= -E_1^{-1} (G_2^{\dagger} X + K_{12}^{\dagger} H_1). \end{aligned} \quad (29)$$

C. Riccati Equations in the Physical Realizability of Linear Quantum Systems

In the coherent quantum H^{∞} control problem considered above (see also, [39], [43], [69], [73] as well in other coherent quantum control problems such as the coherent quantum LQG problem [40]), it is required that the controller be a physically realizable quantum system. One way to achieve this is to first design a classical linear controller and then to add additional quantum noises to the controller to make it physically realizable; e.g., see [39], [69], [74]. In this section, we consider a result from [74] in which the Riccati equation plays a key role in this problem when it is desired to minimize the number of added quantum noises to make a system physically realizable.

Here, we consider quantum linear systems described by QSDEs of the following form (see [39]):

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t); \\ dy(t) &= Cx(t)dt + Ddw(t). \end{aligned} \quad (30)$$

Here, $x(t) = [x_1(t) \cdots x_n(t)]^T$ is a column vector of n self-adjoint system variables which are operators on the underlying Hilbert space. The components of vector x are either position operators q_i or momentum operators p_i , which are related to the annihilation and creation operators considered above as follows:

$$q_i = a_i + a_i^*; \quad p_i = -\imath a_i + \imath a_i^*.$$

Similarly, $dw(t)$ is a column vector of n_w self-adjoint, non-commutative operators representing the input to the system and $dy(t)$ is a column vector of n_y self-adjoint, non-commutative operators representing the output of the system. The components of vector w are either input field position operators Q_i or input field momentum operators P_i , which are related to the input field annihilation and creation operators considered above as follows:

$$Q_i = \mathcal{A}_i + \mathcal{A}_i^*; \quad P_i = -\imath \mathcal{A}_i + \imath \mathcal{A}_i^*.$$

Also, the components of the vector y are either output field position operators Q_i^{out} or output field momentum operators P_i^{out} , which are related to the output field annihilation and creation operators considered above as follows:

$$Q_i^{out} = \mathcal{A}_i^{out} + \mathcal{A}_i^{out*}; \quad P_i^{out} = -\imath \mathcal{A}_i^{out} + \imath \mathcal{A}_i^{out*}.$$

We consider the problem of implementing an arbitrary, strictly proper, LTI system as a quantum system (for example when implementing a coherent controller) by introducing vacuum noise sources. The resulting quantum systems are described by the following QSDEs which are a special case of (30):

$$\begin{aligned} dx(t) &= Ax(t)dt + B_u du(t) \\ &\quad + B_{v_1} dv_1(t) + B_{v_2} dv_2(t); \\ dy(t) &= Cx(t)dt + dv_1(t). \end{aligned} \quad (31)$$

Here, $u(t)$ (a column vector with n_u components) represents the inputs to the system. Also, $v_1(t)$ and $v_2(t)$ (column vectors with n_{v_1} and n_{v_2} components respectively) are quantum Wiener processes corresponding to the introduced vacuum noise inputs. For convenience, the vacuum noises are partitioned into two vectors $v_1(t)$ and $v_2(t)$ such that $n_{v_1} = n_u$. Then, $n_v = n_{v_1} + n_{v_2}$ is the total number of introduced vacuum noise inputs. Subsequently, we will refer to v_1 as the *direct feedthrough quantum noises* and to v_2 as the *additional quantum noises*.

Definition 3: The system described by (30) is *physically realizable* if there exists a quadratic Hamiltonian operator $\mathcal{H} = \frac{1}{2}x(0)^T R x(0)$, where R is a real, symmetric, $n \times n$ matrix, and a coupling operator vector $\mathcal{L} = \Lambda x(0)$, where Λ is a complex-valued $\frac{1}{2}n_w \times n$ coupling matrix such that the matrices A , B , C and D are given by:

$$A = 2\Theta (R + \Im(\Lambda^{\dagger} \Lambda)); \quad (32a)$$

$$B = 2i\Theta [-\Lambda^{\dagger} \quad \Lambda^T] \Gamma; \quad (32b)$$

$$C = P^T \begin{bmatrix} \Sigma_{n_y} & 0 \\ 0 & \Sigma_{n_y} \end{bmatrix} \begin{bmatrix} \Lambda + \Lambda^{\#} \\ -i\Lambda + i\Lambda^{\#} \end{bmatrix}; \quad (32c)$$

$$D = [I_{n_y \times n_y} \quad 0_{n_y \times (n_w - n_y)}]. \quad (32d)$$

Here:

$$\Theta = \begin{bmatrix} J & 0 & \cdots & 0 \\ 0 & J & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J \end{bmatrix}; \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (33)$$

$\Gamma_{n_w \times n_w} = P \text{diag}(M)$; $M = \frac{1}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$; $\Sigma_{n_y} = \begin{bmatrix} I_{\frac{1}{2}n_y \times \frac{1}{2}n_y} & 0_{\frac{1}{2}n_y \times \frac{1}{2}(n_w - n_y)} \end{bmatrix}$; P is the appropriately dimensioned square permutation matrix such that $P \begin{bmatrix} a_1 & a_2 & \cdots & a_{2m} \end{bmatrix} = \begin{bmatrix} a_1 & a_3 & \cdots & a_{2m-1} & a_2 & a_4 & \cdots & a_{2m} \end{bmatrix}$ and $\text{diag}(M)$ is an appropriately dimensioned square block diagonal matrix with each diagonal block equal to the matrix M . (Note that the dimensions of P and $\text{diag}(M)$ can always be determined from the context in which they appear.) $\Im(\cdot)$ denotes the imaginary part of a matrix and † denotes the complex conjugate transpose of a matrix.

The following theorem shows how a Riccati equation with a skew-symmetric solution can be used to determine if a given transfer function can be physically realized with only direct feedthrough noises and no additional quantum noises.

Theorem 6: Consider a system with strictly proper transfer function matrix:

$$G(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}_u.$$

Suppose the algebraic Riccati equation (ARE)

$$X\tilde{B}_u\Theta_{n_u}\tilde{B}_u^T X - \tilde{A}^T X - X\tilde{A} - \tilde{C}^T \Theta_{n_y} \tilde{C} = 0 \quad (34)$$

has a non-singular, real, skew-symmetric solution X . Here, the matrices Θ_{n_u} and Θ_{n_y} are defined as in (33). Then there exists matrices $\{A, B_u, C\}$ such that

$$G(s) = C(sI - A)^{-1}B_u$$

and the corresponding system (31) is physically realizable with only the direct feedthrough quantum noises v_1 and no additional quantum noises v_2 .

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